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On rigidity and realizability of weighted graphs ☆

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Abstract

Recently, a new characterization of rigid graphs was introduced using Euclidean distance matrices (EDMs) [A.Y. Alfakih, Linear Algebra Appl. 310 (2000) 149]. In this paper, we address the computational aspects of this characterization. Also we present a characterization of graphs which are realizable in $\mathbb{R}^r$ for some $1 \leqslant r \leqslant n-2$ but not realizable in $\mathbb{R}^{(n-1)}$, where n is the number of nodes. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $G = (V, E, \omega)$ be an incomplete edge-weighted simple graph where $V = \{v_1, v_2, \ldots, v_n\}$ is the node set and $E \subset V \times V$ is the edge set such that each edge $(v_i, v_j) \in E$ has a positive weight ω_{ij}. A realization of G in $\mathbb{R}^r$ is a mapping of the vertices of G into points in $\mathbb{R}^r$ such that every two adjacent vertices v_i, v_j of G are mapped into points $p^i, p^j \in \mathbb{R}^r$ whose Euclidean distance is equal to the weight ω_{ij}. For the purposes of this paper, it is convenient to view each edge (v_i, v_j) of G as a rigid bar of length ω_{ij} which can rotate freely around its end nodes. Then the graph

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rigidity problem is the problem of determining whether or not the resulting structure of bars and joints of a given realization in $\mathbb{R}^r$ of a weighted graph G is rigid. This problem is well known and has been studied by many authors for well over a century. For an excellent introduction to the subject see [3,5,8,9,11].

Recently [1], a new characterization of rigid realizations of weighted graphs was introduced using Euclidean distance matrices (EDM). In particular, it was shown in [1] that each realization of a given weighted graph G can be represented as a point in a convex compact set $\Omega \subset \mathbb{R}^{\bar{m}}$, where $\bar{m}$ is the number of missing edges in G; and that a generic realization of graph G of n vertices, which admits a realization in $\mathbb{R}^{(n-1)}$, is rigid if and only if its corresponding point is a vertex of Ω. A vertex of Ω is an extreme point of Ω whose normal cone is full dimensional. Thus, for any point $y \in \Omega$, the rank of an associated matrix $C(y)$ can be computed to determine whether y is a vertex of Ω or not. Unfortunately, $C(y)$ as presented in [1] is dense and its entries are generally not rational. In this paper, we show that $C(y)$ can be reduced to a much simpler and relatively sparse matrix C' whose entries are easy to compute and are rational whenever the given realization is rational. Furthermore, the size of C' is polynomially bounded by the size of the realization matrix. Thus, computing C' and its rank can be done efficiently.

If a weighted graph G of n vertices has a realization in $\mathbb{R}^r$, then r must be $\leqslant n-1$. This follows since $r = n-1$ corresponds to the case where the vertices of G are mapped into a set of affinely independent vectors. In [1], it was assumed that graphs satisfy the property that a weighted graph has a realization in $\mathbb{R}^{(n-1)}$ whenever it has a realization in $\mathbb{R}^r$ for some $1 \leqslant r \leqslant n-2$. In this paper, we present a characterization of graphs that do not satisfy this property. A class of such graphs is also presented as an illustration of this characterization.

The paper is organized as follows. In Section 1, we introduce some preliminary definitions and notation. In Section 2, we briefly review the properties of the set Ω, obtained in [1], which are needed for this paper. In the first part of Section 3 we present the main rigidity results from [1] while the computational aspects of rigidity are discussed in the second part of Section 3. The realizability of weighted graphs in $\mathbb{R}^{(n-1)}$ is discussed in Section 4. The paper concludes with a summary in Section 5.

1.1. *Preliminary definitions*

Given an edge-weighted connected graph $G = (V, E, \omega)$ with n nodes and m edges and points $p^1, p^2, \ldots, p^n \in \mathbb{R}^r$, the $n \times r$ matrix

$$P := \begin{bmatrix} {p^1}^{\mathrm{T}} \\ {p^2}^{\mathrm{T}} \\ \vdots \\ {p^n}^{\mathrm{T}} \end{bmatrix}$$

is called a *realization* of G in $\mathbb{R}^r$, or an *r-realization*, if

$$\left\| p^i - p^j \right\| = \omega_{ij} \quad \text{for all} \ \ (v_i, v_j) \in E,$$

$\| \cdot \|$ denotes the Euclidean norm. A weighted graph G is said to be *r-realizable* if there exists a realization of G in $\mathbb{R}^r$. Let P be a realization of G. Then the $n \times n$ matrix $D_P = (d_{ij}) = \| p^i - p^j \|^2$ is called the EDM associated with P. Clearly, D_P is symmetric with zero diagonal and $d_{ij} = \omega_{ij}^2$ for all $(v_i, v_j) \in E$.

Two realizations P and P' of $G = (V, E, \omega)$ are said to be *congruent* if $D_P = D_{P'}$. Note that D_P is invariant under translations, rotations, and space inversion in $\mathbb{R}^r$. Thus, all realizations P' obtained from P by any of the above transformations are congruent to P. In order to avoid trivialities arising from such transformations, no distinction between congruent realizations will be made. Consequently, we can assume that in all realizations P of G, the centroid of the points $p^1, p^2, \ldots, p^n$ coincides with the origin, i.e., $P^{\mathrm{T}} e = 0$, where $e \in \mathbb{R}^n$ is the vector of all 1's.

The pair (G, P) consisting of a weighted graph G and its realization P in $\mathbb{R}^r$ is called a *framework*. A framework (G, P) is generic if P is a generic realization of G. The framework (G, P) is said to be flexible if there exists a differentiable function $\gamma(t)$: $t \in [0, 1] \rightarrow \mathbb{R}^{n \times r}$ such that $\gamma(0) = P$, and $\gamma(t)$ is a realization of G, noncongruent to P, for all t, $0 < t \leqslant 1$. Such a path γ is called a *flexing* of (G, P). The framework (G, P) is said to be *rigid* if it is not flexible.

Vectors $p^1, p^2, \ldots, p^n$ are said to be *affinely independent iff* $\lambda_k = 0$ for $k = 1, \ldots, n$ is the only solution of the system of equations

$$\sum_{k=1}^{n} \lambda_k p^k = 0, \qquad \sum_{k=1}^{n} \lambda_k = 0.$$

Finally, given a graph $G = (V, E)$, $\mathscr{C} \subset V$ is said to be a *clique iff* $(v_i, v_j) \in E$ for every $v_i, v_j \in \mathscr{C}$.

1.2. Notation

We denote by $\mathscr{S}_n$ the space of $n \times n$ symmetric matrices. The inner product on $\mathscr{S}_n$ is given by

$$\langle A, B \rangle := \operatorname{trace}(AB).$$

$B \in \mathscr{S}_n$ is said to be positive semidefinite, denoted by $\succeq 0$, if all of its eigenvalues are nonnegative; and it is said to be positive definite, denoted by $\succ 0$, if all of its eigenvalues are positive.

We denote by e the vector, of the appropriate dimension, of all 1's; and by E^{ij} the symmetric matrix, of the appropriate dimension, with 1's in the (i, j)th and (j, i)th entries and 0's elsewhere. The $n \times n$ identity matrix will be denoted by I_n. $A \circ B$ denotes the *Hadamard*, or the element-wise, product of matrices A and B. The diagonal of a matrix A will be denoted by diag A. The null space of a matrix X is denoted by null X; and we denote by W and U the matrices whose columns form an orthonormal basis for the range space and the null space of X, respectively. 'Starred' matrices refer to a given specific realization of a weighted graph G. Finally, the set theoretic difference will be denoted by '\'.

2. Ω, the set of realizations of graph G

In this section, we briefly review the main results in [1] concerning the set Ω of all realizations of a weighted graph G. Let $G = (V, E, \omega)$ be a given edge-weighted graph with n vertices and m edges. Given a realization P of a graph G in $\mathbb{R}^r$, let $B := PP^{\mathrm{T}}$. Then, B is an $n \times n$ positive semidefinite matrix of rank r. Furthermore, $Be = 0$ since we assume that $P^{\mathrm{T}}e = 0$. Recall that e is the vector of all 1's in $\mathbb{R}^n$.

Let V be an $n \times (n-1)$ matrix whose columns form an orthonormal basis for the orthogonal complement of e, i.e., V satisfies

$$V^{\mathrm{T}}e = 0, \qquad V^{\mathrm{T}}V = I_{n-1}. \tag{1}$$

Let $X := V^{\mathrm{T}}BV$. Then it was shown in [1] that any one of the matrices P, B or X uniquely determines the other two. Hence, we have the following theorem.

Theorem 2.1 [1]. *Given a realization P of a weighted graph $G = (V, E, \omega)$ such that $P^{\mathrm{T}}e = 0$. Then realization P can be equivalently represented by the matrices B or X, where $B := PP^{\mathrm{T}}$ and $X := V^{\mathrm{T}}BV$.*

Thus, a realization in $\mathbb{R}^r$ of a weighted graph G can be equivalently represented by the $n \times r$ matrix P, by the $n \times n$ positive semidefinite matrix B of rank r such that $Be = 0$ or by the $(n-1) \times (n-1)$ positive semidefinite matrix X of rank r. Therefore, by a slight abuse of notation, the term 'realization' will be used to refer to matrices B and X in addition to P.

In the sequel, we will use the $(n-1) \times (n-1)$ positive semidefinite matrix X as the representation of a given realization of a graph G. Such a representation is the most convenient for our purposes for two reasons. First, it will allow a much simpler characterization of Ω, the set of all realizations of graph G in all dimensional Euclidean spaces, which will be defined in (7). Second, in this representation, the set Ω has the desired property of being full dimensional, whenever the given graph G has a realization in $\mathbb{R}^{(n-1)}$.

Let $H = (h_{ij})$ be the adjacency matrix of graph G. Define the linear operator $\mathscr{A}(X)$ such that

$$\mathscr{A}(X) := H \circ \mathscr{K}_V(X), \tag{2}$$

where '$\circ$' denotes the Hadamard product and

$$\mathscr{K}_V(X) := \operatorname{diag}\bigl(VXV^{\mathrm{T}}\bigr)e^{\mathrm{T}} + e\,\operatorname{diag}\bigl(VXV^{\mathrm{T}}\bigr)^{\mathrm{T}} - 2VXV^{\mathrm{T}}, \tag{3}$$

$\mathscr{K}_V(X)$ maps X to the corresponding EDM. Let $\bar{m}$ be the number of missing edges of G, i.e., $\bar{m} = n(n-1)/2 - m$ and let $E^{ij} \in \mathscr{S}_n$ be the matrix with 1's in the (i, j)th and the (j,i)th entries and 0's elsewhere. For each $h_{ij}=0$ and $i<j$, define the matrices M^k, $k=1,\ldots,\bar{m}$, such that

$$M^k := -\frac{1}{2}V^{\mathrm{T}}E^{ij}V. \tag{4}$$

Then it is easy to show that $\{M^k\colon\ k=1,\ldots,\bar{m}\}$ is a set of linearly independent matrices; and that $\{M^k\colon\ k=1,\ldots,\bar{m}\}$ forms a basis for null $\mathscr{A}$ (see [1]). Thus,

$$\text{Null}\ \mathscr{A} = \left\{B\in\mathscr{S}_{n-1}\colon\ B=\sum_{k=1}^{\bar{m}} y_k M^k \text{ for some } y\in\mathbb{R}^{\bar{m}}\right\}. \tag{5}$$

In the sequel, we will denote the given r-realization of G whose rigidity or flexibility is being investigated by X^* or P^*. Recall that $X^*=V^{\mathrm{T}}B^*V=V^{\mathrm{T}}P^*P^{*\mathrm{T}}V$ and that $X^*\in\mathscr{S}_{n-1}$ is positive semidefinite with rank r.

Let Ω_r be the subset of $\mathbb{R}^{\bar{m}}$ defined by

$$\Omega_r := \left\{y\in\mathbb{R}^{\bar{m}}\colon\ X(y) := X^*+\sum_{k=1}^{\bar{m}} y_k M^k \succeq 0,\ \text{and}\ \ \text{rank}\ X(y)=r\right\}. \tag{6}$$

Then we have the following theorem:

Theorem 2.2 [1]. *Given a realization X^* of a weighted graph G in $\mathbb{R}^r$, then $\{X(y)\colon\ y\in\Omega_r\}$ is the set of all realizations of G in $\mathbb{R}^r$, where Ω_r is as defined in* (6).

By relaxing the rank condition on $X(y)$ in (6) we get the following closed, convex, and generally nonpolyhedral set.

$$\Omega := \left\{y\in\mathbb{R}^{\bar{m}}\colon\ X(y)=X^*+\sum_{k=1}^{\bar{m}} y_k M^k \succeq 0\right\}. \tag{7}$$

It readily follows that $\{X(y)\colon\ y\in\Omega\}$ is the set of all realizations of G in all Euclidean spaces of dimension $\leqslant n-1$. Since $X^*\succeq 0$, the origin is always contained in Ω. Furthermore, it was shown in [1] that Ω is bounded if and only if the given graph G is connected.

3. Rigidity of realizations of weighted graphs

In this section, we address the rigidity problem of graphs. First we review the characterization of rigid graphs given in [1]. Then we address the computational aspect of this characterization in Section 3.1.

A point $\hat{y}\in\Omega$ is said to be an *extreme point* of Ω if $\hat{y}$ cannot be represented as a proper convex combination of two distinct points y^1 and y^2 in Ω. Let $\hat{y}$ be an extreme point of Ω. Then the *normal cone* of $\hat{y}$, denoted by $N(\Omega,\hat{y})$, is defined by

$$N(\Omega, \hat{y}) := \left\{c \in \mathbb{R}^{\bar{m}}:\ c^{\mathrm{T}}\hat{y} \geqslant c^{\mathrm{T}}y \text{ for all } y \in \Omega\right\}. \tag{8}$$

An extreme point $\hat{y}$ of Ω is said to be a *vertex* of Ω if $N(\Omega, \hat{y})$ is full dimensional. The following theorem is the main result of [1], namely, Corollary 4.2. Here we explicitly state the assumption made in [1] that the graph is $(n-1)$-realizable.

Theorem 3.1. *Let X^* be a generic realization of a weighted graph G, which admits a realization in $\mathbb{R}^{(n-1)}$. Then the generic framework (G, X^*) is rigid if and only if $y = 0$ is a vertex of Ω.*

Recall from [1] that the set Ω is full dimensional whenever G admits a realization in $\mathbb{R}^{(n-1)}$. A characterization of the normal cone $N(\Omega, \hat{y})$ is given in the following lemma.

Lemma 3.1 [1]. *Assume Ω is full dimensional. Let $\hat{y} \in \Omega$ such that rank $X(\hat{y}) = r$ and let $\bar{r} = (n-1) - r$. Then the normal cone $N(\Omega, \hat{y})$ is given by*

$$N(\Omega, \hat{y}) = \left\{c \in \mathbb{R}^{\bar{m}}:\ c_k = -\left\langle M^k, U\Psi U^{\mathrm{T}}\right\rangle \text{ for some } \Psi \succeq 0\right\}, \tag{9}$$

where U is the $(n-1) \times \bar{r}$ matrix whose columns form an orthonormal basis of the null space of $X(\hat{y})$, and matrix Ψ is $\bar{r} \times \bar{r}$.

From (9) it easily follows that the columns of the following $\bar{m} \times \bar{r}(\bar{r}+1)/2$ matrix $C(0)$ span the normal cone $N(\Omega, 0)$.

$$C(0) = -\begin{bmatrix} u_1^{\mathrm{T}}M^1u_1 & \cdots & u_i^{\mathrm{T}}M^1u_j & \cdots & u_{\bar{r}}^{\mathrm{T}}M^1u_{\bar{r}} \\ u_1^{\mathrm{T}}M^2u_1 & \cdots & u_i^{\mathrm{T}}M^2u_j & \cdots & u_{\bar{r}}^{\mathrm{T}}M^2u_{\bar{r}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_1^{\mathrm{T}}M^{\bar{m}}u_1 & \cdots & u_i^{\mathrm{T}}M^{\bar{m}}u_j & \cdots & u_{\bar{r}}^{\mathrm{T}}M^{\bar{m}}u_{\bar{r}} \end{bmatrix}, \tag{10}$$

where $i = 1, \ldots, \bar{r}$, $j = i, \ldots, \bar{r}$, and u_i is the ith column of U. U is the $(n-1) \times \bar{r}$ matrix whose columns form an orthonormal basis of the null space of X^*. Therefore, a given generic realization X^* in $\mathbb{R}^r$ of weighted graph G, which admits a realization in $\mathbb{R}^{(n-1)}$, is rigid if and only if the rank of $C(0) = \bar{m}$.

3.1. Computational issues

Note that $C(0)$ in (10) is a dense matrix whose entries need not be rational. In this section, we show that $C(0)$ can be reduced to a much simpler matrix C' where C' is rational and relatively sparse.

Let P^* be the r-realization, for some positive integer $r \leqslant n-1$, of $G = (V, E, \omega)$ whose rigidity or flexibility is being investigated. Let Λ be an $n \times \bar{r}$ matrix, $\bar{r} = n - 1 - r$, whose columns form a basis for the null space of the $(r+1) \times n$ matrix

$$\begin{bmatrix} P^{*\mathrm{T}} \\ e^{\mathrm{T}} \end{bmatrix},$$

that is,

$$P^{*\mathrm{T}}\Lambda = 0, \quad e^{\mathrm{T}}\Lambda = 0 \text{ and the columns of } \Lambda \text{ are linearly independent.} \tag{11}$$

Thus, Λ is rational whenever P^* is rational. Note that the columns of Λ represent the affine dependence relationships among the points $p^{*1}, p^{*2}, \ldots, p^{*n}$, i.e., the rows of P^*. The following lemma establishes the relationship between Λ and matrices V and U.

Lemma 3.2. *Let Λ be the matrix defined in* (11). *Then there exists a nonsingular $\bar{r} \times \bar{r}$ matrix Q such that $\Lambda Q = VU$, where V was defined in* (1) *and U is the matrix whose columns form an orthonormal basis of null space of X^*.*

Proof. Recall that $X^* = V^{\mathrm{T}}P^*P^{*\mathrm{T}}V$. Then from the definition of U we have $X^*U = 0$. This implies that $P^{*\mathrm{T}}VU = 0$. Furthermore, $e^{\mathrm{T}}V = 0$. Hence, the columns of VU form an orthonormal basis for the null space of

$$\begin{bmatrix} P^{*\mathrm{T}} \\ e^{\mathrm{T}} \end{bmatrix}.$$

Therefore, there exists an $\bar{r} \times \bar{r}$ nonsingular matrix Q such that $\Lambda Q = VU$ since the columns of Λ and VU form bases for the same space. $\square$

Thus from (4) and Lemma 3.2 it follows that

$$-\langle M^k, U\Psi U^{\mathrm{T}}\rangle = 1/2\langle \Lambda^{\mathrm{T}}E^{ij}\Lambda, Q\Psi Q^{\mathrm{T}}\rangle.$$

Therefore, the normal cone $N(\Omega, 0)$ is equivalently given by

$$N(\Omega, 0) = \{c \in \mathbb{R}^{\bar{m}}\colon\ c_k = \langle \Lambda^{\mathrm{T}}E^{ij}\Lambda, \Psi\rangle \text{ for some } \Psi \succeq 0\}, \tag{12}$$

since $\Psi \succeq 0$ if and only if $Q\Psi Q^{\mathrm{T}} \succeq 0$. Note that the indices i and j in (12) run over all $(v_i, v_j) \notin E$.

Since the columns of Λ are linearly independent, it follows that rank $\Lambda = \bar{r}$. By relabeling the nodes if necessary, we can assume without loss of generality that

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix},$$

where Λ_1 is a nonsingular $\bar{r} \times \bar{r}$ matrix. Then define the matrix Z by

$$Z := \Lambda\Lambda_1^{-1} = \begin{bmatrix} I_{\bar{r}} \\ \Lambda_2\Lambda_1^{-1} \end{bmatrix}. \tag{13}$$

Now since $\langle \Lambda^{\mathrm{T}}E^{ij}\Lambda, \Psi\rangle = \langle Z^{\mathrm{T}}E^{ij}Z, \Lambda_1\Psi\Lambda_1^{\mathrm{T}}\rangle$, it follows that the normal cone $N(\Omega, 0)$ is given by

$$N(\Omega, 0) = \{c \in \mathbb{R}^{\bar{m}}\colon\ c_k = \langle z^i z^{j\mathrm{T}} + z^j z^{i\mathrm{T}}, \Psi\rangle \text{ for some } \Psi \succeq 0\}, \tag{14}$$

where z^i is the ith row of Z.

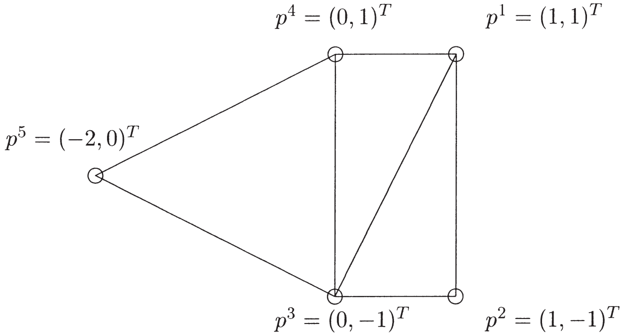

Fig. 1. Realization P^* in $\mathbb{R}^2$ of weighted graph G in Example 1.

Let us label the missing edges of G as $(v_{i_1}, v_{j_1}), (v_{i_2}, v_{j_2}), \ldots, (v_{i_{\bar m}}, v_{j_{\bar m}})$. Then the columns of the following $\bar m \times \bar r(\bar r + 1)/2$ matrix C' span $N(\Omega, 0)$.

$$C' = \begin{bmatrix} 2z_{i_1 1}z_{j_1 1} & \cdots & z_{i_1 p}z_{j_1 q} + z_{j_1 p}z_{i_1 q} & \cdots & 2z_{i_1 \bar r}z_{j_1 \bar r} \\ 2z_{i_2 1}z_{j_2 1} & \cdots & z_{i_2 p}z_{j_2 q} + z_{j_2 p}z_{i_2 q} & \cdots & 2z_{i_2 \bar r}z_{j_2 \bar r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2z_{i_{\bar m} 1}z_{j_{\bar m} 1} & \cdots & z_{i_{\bar m} p}z_{j_{\bar m} q} + z_{j_{\bar m} p}z_{i_{\bar m} q} & \cdots & 2z_{i_{\bar m} \bar r}z_{j_{\bar m} \bar r} \end{bmatrix}, \tag{15}$$

where $p = 1, 2, \ldots, \bar r$ and $q = p, \ldots, \bar r$. Therefore, $N(\Omega, 0)$ is full dimensional *iff* rank $C' = \bar m$. Note that from (13) it follows that Z has at least $\bar r(\bar r - 1)$ zero entries. Thus, many entries of C' would be 0's.

Let s be a rational number such that $s = p/q$, where p is an integer, q a natural number, and p and q are relatively prime. Then define

$$\text{size}(s) := 1 + \lceil \log_2(|p| + 1) \rceil + \lceil \log_2(q + 1) \rceil.$$

Let $A = (a_{ij})$ be an $n \times n$ nonsingular rational matrix. Define size of $A := n^2 + \sum_{ij} \text{size}(a_{ij})$. Then the size of A^{-1} is polynomially bounded by the size of A (see [10, p. 30, Corollaries 3.2a and 3.2c]). Assuming that the matrix P^* is rational, it follows from (11), (13), and (15) that the size of C' is polynomially bounded by the size of P^*. Therefore, the rank of C' can be computed in polynomial time.

As an illustration we present a numerical example.

3.2. *Example 1*

Consider the realization of graph G in $\mathbb{R}^2$ given in Fig. 1. In this case, $n = 5$, $r = 2$, $\bar r = 2$, and $\bar m = 3$. Furthermore,

$$P^* = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & -1 \\ 0 & 1 \\ -2 & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -1 & -1/2 \\ 1 & 5/2 \\ -1 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1/4 & -5/4 \\ -5/4 & -1/4 \\ 1/2 & 1/2 \end{bmatrix}.$$

The missing edges are: (v_1, v_5), (v_2, v_4), (v_2, v_5). Thus, the matrix C' is 3×3 and it is given by

$$C' = \begin{bmatrix} 2z_{11}z_{51} & z_{11}z_{52} + z_{51}z_{12} & 2z_{12}z_{52} \\ 2z_{21}z_{41} & z_{21}z_{42} + z_{41}z_{22} & 2z_{22}z_{42} \\ 2z_{21}z_{51} & z_{21}z_{52} + z_{51}z_{22} & 2z_{22}z_{52} \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & -5/4 & -1/2 \\ 0 & 1/2 & 1 \end{bmatrix}.$$

Clearly rank $C' = 3$. Thus, $N(\Omega, 0)$ is full dimensional and the given realization P^* is rigid.

4. Realizations of weighted graphs in $\mathbb{R}^{(n-1)}$

In [1] it was assumed that graphs satisfy the property that a weighted graph is $(n-1)$-realizable whenever it is r-realizable for some positive integer $r \leqslant n-2$. In this section, we present a characterization of weighted graphs which do not satisfy this property. We will denote by X^* or P^* the given realization of graph G in $\mathbb{R}^r$.

Lemma 4.1. *Let X^* be a given realization of a weighted graph $G = (V, E, \omega)$ in $\mathbb{R}^r$ for some positive integer $r \leqslant n-1$. Then G is $(n-1)$-realizable if and only if there exists $\hat{y} \in \mathbb{R}^{\bar{m}}$ such that*

$$X^* + \sum_{k=1}^{\bar{m}} \hat{y}_k M^k \succ 0.$$

Proof. If there exists $\hat{y} \in \mathbb{R}^{\bar{m}}$ such that $X(\hat{y}) = X^* + \sum_{k=1}^{\bar{m}} \hat{y}_k M^k \succ 0$, then rank $X(\hat{y}) = n-1$. Hence, $X(\hat{y})$ is a realization of G in $\mathbb{R}^{n-1}$. To show the other direction, assume that G is $(n-1)$-realizable. Then there exists a realization X of G such that $X \succeq 0$ and rank $X = n-1$. But since X is an $(n-1) \times (n-1)$ matrix it follows that X must be positive definite. On the other hand, since X is a realization of G, $X - X^*$ must belong to the null space of $\mathscr{A}(\cdot) = H \circ \mathscr{K}_V(\cdot)$. Hence, from (5) there exists $\hat{y} \in \mathbb{R}^{\bar{m}}$ such that $X - X^* = \sum_{k=1}^{\bar{m}} \hat{y}_k M^k$. $\square$

The following lemma is a known result which holds for any general solid convex cone [2,4,6,7]. It is stated here for the special case of the cone of positive semidefinite matrices. For completeness we present a proof of this lemma.

Lemma 4.2. *Given matrices A^k, $k = 0, 1, \ldots, \bar{m}$, in $\mathscr{S}_{n-1}$, exactly one of the following two systems has a solution*:

(i) $A^0 + \sum_{k=1}^{\bar{m}} y_k A^k \succ 0$;

(ii) $Y \succeq 0$, $Y \neq 0$, $\langle A^0, Y \rangle \leqslant 0$, *and* $\langle A^k, Y \rangle = 0$ *for* $k = 1, \ldots, \bar{m}$.

Proof. If both systems (i) and (ii) have solutions $y \in \mathbb{R}^{\bar{m}}$ and $Y \succeq 0$ $Y \neq 0$, respectively, then $0 < \langle Y, A^0 + \sum_{k=1}^{\bar{m}} y_k A^k \rangle = \langle Y, A^0 \rangle + \sum_{k=1}^{\bar{m}} \langle Y, A^k \rangle = \langle Y, A^0 \rangle \leqslant 0$; a contradiction.

Now assume that system (i) has no solution. Let $X(y) := A^0 + \sum_{k=1}^{\bar{m}} y_k A^k$ and let

$$\mathscr{L} := \left\{ B \in \mathscr{S}_{n-1}\colon\ B = C - X(y) \text{ for some } C \succ 0 \text{ and some } y \in \mathbb{R}^{\bar{m}} \right\}.$$

Then $0 \notin \mathscr{L}$. Furthermore, $\mathscr{L}$ is an open convex set. Hence, by separation theorem there exists $Y \neq 0$ such that $\langle Y, B \rangle \geqslant 0$ for all $B \in \mathscr{L}$. Given some $\hat{C} \succ 0$ and $X(\hat{y})$ for some $\hat{y} \in \mathbb{R}^{\bar{m}}$, then for a sufficiently large scalar λ we have $\hat{C} + \lambda^{-1} X(\hat{y}) \succ 0$. Hence, $\lambda \hat{C} \in \mathscr{L}$, consequently $\langle Y, \lambda \hat{C} \rangle \geqslant 0$. Thus, $\langle Y, \hat{C} \rangle \geqslant 0$ and $Y \succeq 0$. On the other hand, $\epsilon \hat{C} - X(\hat{y}) \in \mathscr{L}$ for all $\epsilon > 0$. Therefore, $\langle Y, -X(\hat{y}) \rangle \geqslant 0$ as $\epsilon \downarrow 0$. Hence, $\langle A^0, Y \rangle \leqslant 0$, and $\langle A^k, Y \rangle = 0$ for $k = 1, \ldots, \bar{m}$ since $\hat{y}$ was arbitrary. $\square$

Lemma 4.3. *Let* $A^0 \succeq 0$, *and* $Y \succeq 0$ *and let* $\langle A^0, Y \rangle = 0$. *Then* $Y = U \Psi U^{\mathrm{T}}$ *for some* $\Psi \succeq 0$, *where U is the matrix whose columns form an orthonormal basis for the null space of* A^0.

Proof. Let W be the matrix whose columns form an orthonormal basis for the range space of A^0 and let U be the matrix whose columns form an orthonormal basis for the null space of A^0. Thus, the matrix $Q = [W\ U]$ is orthogonal. Since both A^0 and Y are positive semidefinite, it is well known that $\langle A^0, Y \rangle = 0$ if and only if $A^0 Y = 0$. But $A^0 Y = 0$ implies that $\operatorname{diag}(W^{\mathrm{T}} Y W) = 0$, and since $W^{\mathrm{T}} Y W \succeq 0$, it also implies that $W^{\mathrm{T}} Y W = 0$. Furthermore, since

$$Q^{\mathrm{T}} Y Q = \begin{bmatrix} W^{\mathrm{T}} Y W & W^{\mathrm{T}} Y U \\ U^{\mathrm{T}} Y W & U^{\mathrm{T}} Y U \end{bmatrix} \succeq 0,$$

it follows that $W^{\mathrm{T}} Y U = U^{\mathrm{T}} Y W = 0$. Let $\Psi = U^{\mathrm{T}} Y U$, then

$$Y = [W\ U] \begin{bmatrix} 0 & 0 \\ 0 & \Psi \end{bmatrix} \begin{bmatrix} W^{\mathrm{T}} \\ U^{\mathrm{T}} \end{bmatrix} = U \Psi U^{\mathrm{T}}. \qquad \square$$

System (ii) in Lemma 4.2, in the case of special matrices X^* and M^k, $k = 1, 2, \ldots, \bar{m}$, can be further simplified as shown in the following lemma.

Lemma 4.4. *Given the matrices* $X^*, M^1, \ldots, M^{\bar{m}}$ *defined in* (4), *then the system*

$$Y \succeq 0, \quad Y \neq 0, \quad \textit{and} \quad \left\langle X^*, Y \right\rangle \leqslant 0, \quad \left\langle M^k, Y \right\rangle = 0, \quad k = 1, \ldots, \bar{m},$$

is equivalent to

$$\Psi \succeq 0, \quad \Psi \neq 0, \quad \textit{and} \quad {z^i}^{\mathrm{T}} \Psi z^j = 0 \quad \textit{for all} \ \ (v_i, v_j) \notin E, \tag{16}$$

where $z^i \in \mathbb{R}^{\bar{r}}$ *denotes the* ith *row of the matrix Z defined in* (13).

Proof. First note that since both $X^* \succeq 0$ and $Y \succeq 0$, $\langle X^*, Y \rangle \leqslant 0$ is equivalent to $\langle X^*, Y \rangle = 0$. From Lemma 4.3 we have that $Y = U \Psi U^{\mathrm{T}}$ for some $\bar{r} \times \bar{r}$ positive semidefinite matrix Ψ. Hence, $\langle Y, M^k\rangle = \langle \Psi, U^{\mathrm{T}}M^kU\rangle$. But from (4) and Lemma 3.2 it follows that $\langle \Psi, U^{\mathrm{T}}M^kU\rangle = 0$, $k = 1, \ldots, \bar{m}$, is equivalent to $\langle \Psi, Q^{\mathrm{T}}\Lambda^{\mathrm{T}}E^{ij}\Lambda Q\rangle = 0$ for all $(v_i, v_j) \notin E$. The assertion follows from (13) since $Z^{\mathrm{T}}E^{ij}Z = z^i{z^j}^{\mathrm{T}} + z^j{z^i}^{\mathrm{T}}$ and since the matrix $\Lambda_1 Q\Psi Q^{\mathrm{T}}\Lambda_1^{\mathrm{T}}$ is nonzero and positive semidefinite iff $\Psi \succeq 0$ and $\Psi \neq 0$. □

Next we have our main result in this section.

Theorem 4.1. *Let P^* be an r-realization of $G = (V, E, \omega)$ for some positive integer $r \leqslant n-2$. Then G is not $(n-1)$-realizable iff the following system*

$$\Psi \succeq 0 \quad \text{and} \quad {z^i}^{\mathrm{T}}\Psi z^j = 0 \quad \text{for all} \ (v_i, v_j) \notin E, \tag{17}$$

has a solution $\Psi \neq 0$, where $z^i \in \mathbb{R}^{\bar{r}}$ denotes the ith *row of the matrix Z defined in* (13).

Proof. This follows from Lemmas 4.1, 4.2, and 4.4. □

Therefore, in order to show that a given r-realizable weighted graph G is $(n-1)$-realizable, it suffices to exhibit a $\hat{y}$ such that $X^* + \sum_{k=1}^{\bar{m}} \hat{y}_k M^k \succ 0$. On the other hand, to show that G is not $(n-1)$-realizable it suffices to exhibit a nonzero matrix Ψ satisfying (17). As an illustration, we discuss, next, the class of graphs for which the system of equations (17) has a rank-1 solution Ψ.

4.1. Example 2: graphs corresponding to Ψ of rank 1

Lemma 4.5. *Let P^* be an r-realization of $G = (V, E, \omega)$ for some $r \leqslant n-2$, and let Z be the matrix defined in* (13). *Let $\mathscr{C} \subset V$. Then the set $\{{p^*}^i : v_i \in \mathscr{C}\}$ is affinely dependent if and only if there exists a nonzero $\xi \in \mathbb{R}^{\bar{r}}$ such that $\xi^{\mathrm{T}}z^j = 0$ for all $v_j \in V\backslash\mathscr{C}$, where z^i is the* ith *row of Z.*

Proof. Wlog assume that $\mathscr{C} = \{v_1, v_2, \ldots, v_q\}$. Then, $\{{p^*}^1 {p^*}^2 \cdots {p^*}^q\}$ is affinely dependent iff there exists a nonzero $\hat{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_q)^{\mathrm{T}}$ such that $\sum_{i=1}^{q} \lambda_i {p^*}^i = 0$ and $\sum_{i=1}^{q} \lambda_i = 0$. This is equivalent to $(\hat{\lambda}^{\mathrm{T}} 0 \cdots 0)^{\mathrm{T}}$ belonging to the null space of

$$\begin{bmatrix} P^{*\mathrm{T}} \\ e^{\mathrm{T}} \end{bmatrix}.$$

This is equivalent to the existence of a nonzero $\xi \in \mathbb{R}^{\bar{r}}$ such that

$$\begin{bmatrix} \hat{\lambda} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Z\xi, \tag{18}$$

since the columns of Z form a basis for the null space of

$$\begin{bmatrix} P^{*\mathrm{T}} \\ e^{\mathrm{T}} \end{bmatrix}.$$

The assertion follows since (18) is equivalent to $(0 \cdots 0) = \xi^{\mathrm{T}}[z^{q+1} \cdots z^n]$. $\square$

The following theorem is the main result in this section.

Theorem 4.2. *Let P^* be an r-realization of weighted graph $G = (V, E, \omega)$ for some $r \leqslant n-2$. Then the following two statements are equivalent*:
1. *G has a clique $\mathscr{C}$ such that the set $\{p^{*i}\colon\ v_i \in \mathscr{C}\}$ is affinely dependent.*
2. *G is not $(n-1)$-realizable and the system of equations* (17) *has a solution Ψ of rank* 1.

Proof. Assume that G has a clique $\mathscr{C}$ such that the set $\{p^{*i}\colon\ v_i \in \mathscr{C}\}$ is affinely dependent. Wlog let $\mathscr{C} = \{v_1, v_2, \ldots, v_q\}$. Hence, for every $(v_i, v_j) \notin E$, we must have that either $j \geqslant q+1$ or $i \geqslant q+1$. On the other hand, since the set $\{p^{*i}\colon\ v_i \in \mathscr{C}\}$ is affinely dependent, from Lemma 4.5 there exits a nonzero $\xi \in \mathbb{R}^{\bar{r}}$ such that $\xi^{\mathrm{T}} z^j = 0$ for all $j = q+1, \ldots, n$. Let $\Psi = \xi\xi^{\mathrm{T}}$. Then obviously $\Psi \succeq 0$ and rank $\Psi = 1$. Furthermore,

$$z^{i\,\mathrm{T}} \Psi z^j = \xi^{\mathrm{T}} z^i \;\; \xi^{\mathrm{T}} z^j = 0 \quad \text{for all} \;\; (v_i, v_j) \notin E.$$

Hence Ψ is a solution of (17) and from Theorem 4.1 it follows that G is not $(n-1)$-realizable.

To show the other direction assume that G is not $(n-1)$-realizable and that the system of equations (17) has a solution Ψ of rank 1. Then $\Psi = \xi\xi^{\mathrm{T}}$ for some nonzero $\xi \in \mathbb{R}^{\bar{r}}$ and

$$z^{i\,\mathrm{T}} \Psi z^j = \xi^{\mathrm{T}} z^i \;\; \xi^{\mathrm{T}} z^j = 0 \quad \text{for all} \;\; (v_i, v_j) \notin E. \tag{19}$$

Let $\mathscr{C} = \{v_j \in V\colon\ \xi^{\mathrm{T}} z^j \neq 0\}$. Then Lemma 4.5 implies that the set $\{p^{*i}\colon\ v_i \in \mathscr{C}\}$ is affinely dependent. Furthermore, it follows from (19) that $(v_i, v_j) \in E$ for every $v_i, v_j \in \mathscr{C}$ since otherwise if there exits $v_{i_0}, v_{j_0} \in \mathscr{C}$ such that $(v_{i_0} v_{j_0}) \notin E$, we would have $\xi^{\mathrm{T}} z^{i_0} \;\; \xi^{\mathrm{T}} z^{j_0} \neq 0$ a contradiction of (19). Thus, $\mathscr{C}$ is a clique of G and the set $\{p^{*i}\colon\ v_i \in \mathscr{C}\}$ is affinely dependent. $\square$

5. Summary

This paper is an extension of a recent paper [1]. In [1], it was shown that a given generic realization of a weighted graph, which admits a realization in $\mathbb{R}^{(n-1)}$, is rigid *iff* the origin is vertex of a convex compact set Ω defined in (7). In the first part of this paper, we discussed the computational aspect of characterizing the vertices of Ω.

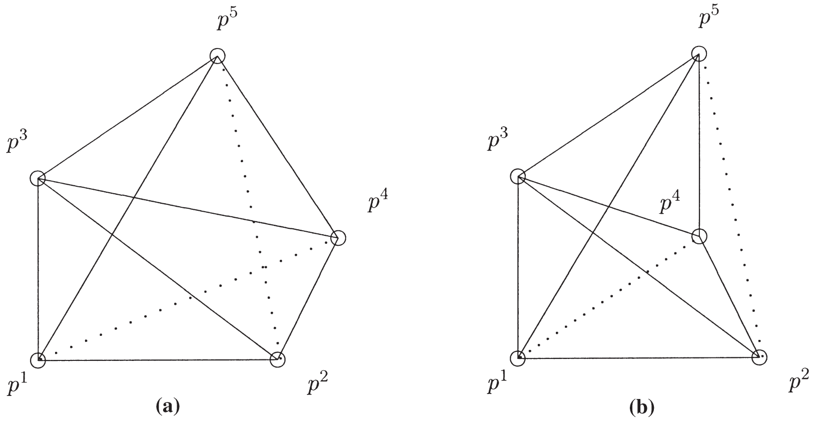

Fig. 2. Realizations in $\mathbb{R}^2$ of two weighted graphs G and G' which are not $(n-1)$-realizable. In this case (17) has a solution Ψ of rank 2. The dotted lines represent missing edges.

In particular, we showed that the origin is a vertex of Ω *iff* the rank of the relatively sparse rational matrix C', defined in (15), is equal to $\bar{m}$.

In the second part of this paper we presented a characterization of weighted graphs that are r-realizable, $1 \leqslant r \leqslant n-2$, but not $(n-1)$-realizable. In particular, we showed that a weighted graph is not $(n-1)$-realizable *iff* the system of equations (17) has a nonzero solution Ψ. As an illustration, we characterized graphs for which (17) has a rank-1 solution Ψ. An example where (17) has a rank-2 solution is given in Fig. 2.

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